$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$$A_{a'} = \frac{\partial x^a}{\partial x^{a'}} A_a \qquad\qquad A^{a'} = \frac{\partial x^{a'}}{\partial x^a} A^a$$

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$= \begin{pmatrix} S_\theta C_\phi & S_\theta S_\phi & C_\theta \\ r\, C_\theta C_\phi & r\, C_\theta S_\phi & -r S_\theta \\ -r S_\theta S_\phi & r S_\theta C_\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

[Note A_r and A_θ, A_ϕ don't have the same dimension]

$$\begin{pmatrix} A^r \\ A^\theta \\ A^\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{pmatrix} \begin{pmatrix} A^x \\ A^y \\ A^z \end{pmatrix}$$

$$= \begin{pmatrix} S_\theta C_\phi & S_\theta S_\phi & C_\theta \\ \frac{1}{r} C_\theta C_\phi & \frac{1}{r} C_\theta S_\phi & -\frac{1}{r} S_\theta \\ -\frac{S_\phi}{r S_\theta} & \frac{C_\phi}{r S_\theta} & 0 \end{pmatrix} \begin{pmatrix} A^x \\ A^y \\ A^z \end{pmatrix}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$\omega = x\,dy - y\,dx =$

$$\begin{pmatrix}\omega_x \\ \omega_y\end{pmatrix} = \begin{pmatrix}-y \\ x\end{pmatrix} \qquad \begin{pmatrix}\omega_r \\ \omega_\theta\end{pmatrix} = \begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta}\end{pmatrix}\begin{pmatrix}\omega_x \\ \omega_y\end{pmatrix}$$

$$\therefore \begin{pmatrix}\omega_r \\ \omega_\theta\end{pmatrix} = \begin{pmatrix}c_\theta & s_\theta \\ -r s_\theta & r c_\theta\end{pmatrix}\begin{pmatrix}-r s_\theta \\ r c_\theta\end{pmatrix} = \begin{pmatrix}0 \\ r^2\end{pmatrix}$$

$$\Rightarrow \quad \omega = r^2\,d\theta$$

$$d\omega = dx\,dy - dy\,dx = 2\,dx\,dy \Rightarrow \omega_{xy} = 1 \quad \omega_{yx} = -1$$

$$= d(r^2 d\theta) = 2r\,dr\,d\theta \Rightarrow \omega_{r\theta} = r \quad \omega_{\theta r} = -r$$

converting forms:

$$\omega_{r\theta} = \frac{\partial x^a}{\partial r}\frac{\partial y^b}{\partial \theta}\,\omega_{ab}$$

$$= \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right) = r \quad \checkmark$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$A\cdot\hat{r} = \sin\theta\cos\phi\,A^x + \sin\theta\sin\phi\,A^y + \cos\theta\,A^z$$

$$= A^r = A_r$$

$$A\cdot\hat{\theta} = \cos\theta\cos\phi\,A^x + \cos\theta\sin\phi\,A^y - \sin\theta\,A^z$$

$$= r\,A^\theta = \frac{1}{r}A_\theta$$

$$A\cdot\hat{\phi} = -\sin\phi\,A^x + \cos\phi\,A^y = r\sin\theta\,A^\phi = \frac{1}{r\sin\theta}A_\phi$$

$$f = r\cos\theta$$

$$df = dr\cos\theta - r\sin\theta\, d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$d(df) = -\sin\theta\, d\theta\, dr - (dr\sin\theta + r\cos\theta\, d\theta)\, d\theta$$

$$= -\sin\theta\, d\theta\, dr - \sin\theta\, dr\, d\theta$$

$$= 0$$

$$A = A_\mu \, dx^\mu$$

$$\therefore \; dA = \frac{\partial A_\mu}{\partial x^\nu} \, dx^\nu \, dx^\mu$$

$$= \frac{1}{2} \left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) dx^\nu \, dx^\mu \qquad \text{since } dx^\mu dx^\nu = - dx^\nu dx^\mu$$

$$= -\frac{1}{2} F_{\mu\nu} \, dx^\nu \, dx^\mu = \frac{1}{2} F_{\mu\nu} \, dx^\mu dx^\nu$$

$$F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \, dx^\nu$$

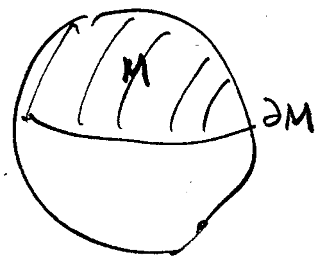
$$0 = dF = \frac{1}{2} \frac{\partial F_{\mu\nu}}{\partial x^\alpha} \, dx^\alpha \, dx^\mu \, dx^\nu$$

$dx^\alpha dx^\mu dx^\nu$ is completely antisymmetric in $(\alpha \, \mu \nu)$

F is antisymmetric in $\mu\nu$.

$$\therefore \; 0 = dF \Rightarrow \frac{\partial F_{\mu\nu}}{\partial x^\alpha} + \frac{\partial F_{\nu\alpha}}{\partial x^\mu} + \frac{\partial F_{\alpha\mu}}{\partial x^\nu} = 0$$

which are the source-free Maxwell eqns.



$$\int_{\partial M} \omega = \int_{\partial M} \sin\theta \, d\phi = 2\pi$$

since $\theta = \pi/2$

$$d\omega = \cos\theta \, d\theta \, d\phi$$

$$\int_M d\omega = \int_M \cos\theta \, d\theta \, d\phi = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \cos\theta$$

$$= 2\pi$$

For $\omega = \cos\theta \, d\phi$ $\quad d\omega = -\sin\theta \, d\theta \, d\phi$

$$\int_M d\omega = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, (-\sin\theta)$$

$$= -2\pi$$

$\int_{\partial M} \omega = 0$ since $\cos \pi/2 = 0$

at $\theta = 0$ $\cos\theta \, d\phi = d\phi$ and is singular since ϕ is not a good coordinate.