1. Compute the components $(\mathcal{L}_X Y)^i$ of the Lie derivative $\mathcal{L}_X$ of the vector field Y.

2. Show that $\sqrt{|g|}\, \mathrm{d}^4 x$ is invariant under a change of coordinates, where $g = \det g_{\mu\nu}$ and $\mathrm{d}^4 x = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

3. Let $x^\mu(s)$ be a curve. Show that $\mathrm{d}x^\mu/\mathrm{d}s$ transforms as a tensor. Find the transformation law for $\mathrm{d}^2 x^\mu/\mathrm{d}s^2$, and show that it does not transform as a tensor.

4. Find the transformation law for $\Gamma^\lambda_{\mu\nu}$ under a change of coordinates:

$$\begin{aligned}
\Gamma^{\lambda'}_{\alpha'\beta'} &= \frac{\partial x^{\lambda'}}{\partial x^\sigma}\frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\nu'}}\Gamma^\sigma_{\mu\nu} - \frac{\partial^2 x^{\lambda'}}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\nu'}} \\
&= \frac{\partial x^{\lambda'}}{\partial x^\sigma}\frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\nu'}}\Gamma^\sigma_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\mu'}\partial x^{\nu'}}
\end{aligned}$$

   using the formula for the connection in terms of derivatives of the metric.

5. Consider the static spherically symmetric metric

$$\mathrm{d}s^2 = e^{A(r)}\left[\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\phi^2\right] - e^{B(r)}\mathrm{d}t^2$$

   Find the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$. You can obtain the result by deriving the geodesic equations using the variational method, or by computing them using the definition in terms of derivatives of the metric.

6. Show that $g_{\alpha\beta}(\mathrm{d}x^\alpha/\mathrm{d}s)(\mathrm{d}x^\beta/\mathrm{d}s)$ is constant along a curve which is a solution of the geodesic equation.

7. Find the equations which extremize

$$\int \mathrm{d}s\ g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}$$